[9] Orlov, S. A. Tsilker, B. Ya. (2012). Technologii razrabotki programmnogo obespechenia. Saint Petersburg: Piter, 608.

[10] Albrecht A. J. (1979). Measuring Application Development Productivity. Proc. IBM Application Development Symposium, 83–92.

INVESTIGATION OF THE STABILITY FOR ESTABLISHED FLOWS IN OPEN PSEUDOPRISMATIC CHANNELS

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Abstract

In this article the equations of channel flow of fluid are considered. They are described by the continuity equation and the momentum equation depending on the rate of flow and cross-sectional area of flow.

It is shown that natural phenomena significantly affect the flow stability and received a condition that ensures the stability of unstated flow at perturbation in initial and boundary conditions. Conditions of stability are checked on test examples, which depend on the following parameters: established movement parameters and middle and bottom slope angles.

It is shown that in such assumptions the system of equations of channel flow of fluid is stable, if received conditions is executed.

Keywords: stability, S-stability, L-stability, channel flow, incompressible fluid, pseudoprismatic channels, perturbations.

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1. Introduction

One of the important tasks for the practice of open-flow stability is investigation of stability in pseudoprismatic channels with big slope angle. The problem is to get the conditions under which the wave that formed in channel as a result of small perturbations will increase or the conditions under which such increase will not.

This problem is different from the usual formulations of stability problems, where deviations are considered at a given point of the flow that will damped during some time.

It should be note that damping of small waves along the channel is a necessary condition but not a sufficient condition for damping waves of finite amplitude, from instability flow at a point follows instable flow in channel, though the stable flow at a point not mean stable flow in general.

Therefore, further along with classical Lyapunov stability, we consider different concepts of stability of open channel flow.

2. Materials and Methods

2. 1. Analysis of the basic phenomena that affect the stability of channel flow

Investigation of the stability of turbulent channel flow [1] associated with the following problems: on the one hand, we can get full information about the stability of the spatial formulation of the hydrodynamics problem; on the other hand, the stability significantly affects the hydrody-

namic pressure, – data, which does not extend beyond the one-dimensional idealization of channel flow. Therefore, this information narrows the limits of stability of turbulent channel flow. In such conditions it is difficult to formulate general conditions for stability, so we present some of the major natural phenomena affecting the stability, namely:

– occurrence of transverse circulation;
– aeration flow;
– occurrence of standing waves;
– occurrence of waves with increasing amplitude.

Transverse circulation in the flow can be explained by the fact that the flow lines of averaged velocities are not straight but spiral. Curvature of the free surface affects the resistance due to deformation of flow lines. In order to account for this effect it should have data on the structure of the velocity field. These data can be obtained from the analysis of the stress tensor structure in a three-dimensional model.

In the channels with a large slope (mountainous and semi-mountainous river) bottom flow may be aerated. Consequently, the output equations describing such flow must proceed from two-phase fluid dynamics. Processes in aerated flows are studied in [2]. The following hypothesis was adopted in the derivation of its equations: the flow has a clear free surface, the viscosity of the liquid inside is not taken into account, etc. Voinovich and Schwartz [3] considered the velocity of waves on the boundary of two media (air and water). The authors obtained a result that indicates the start of aeration conditions, but fails to provide a degree of aeration flow is the most important problem. Trying to explain the origin of aeration, Voinovich and Schwartz was carried out hydrodynamic approach to the space problem. Therefore, our formulation of the problem [4] based on hydrodynamic idealization is too "rough" for this effect.

Moreover, such hydrodynamic formulation of the problem makes it impossible to establish a criterion for occurrence of standing waves, which form an acute angle with the axis of flow. Therefore, they can be described by at least two spatial coordinates.

Therefore, for statement of stability problem of the established channel flow in [5] is possible to analyze the impact of a fourth effect, namely, the occurrence of waves with growing amplitude.

2. 2. Investigation of the stability of wave flows in pseudoprismatic channels

To describe the motion of an incompressible fluid in a pseudoprismatic channel with the vertical plane of symmetry [4–6] using the following system of equations:

$$\frac{\partial Q}{\partial x}+\frac{\partial F}{\partial t}=0, \tag{1}$$

$$\frac{1}{g}\frac{1}{F}\frac{\partial Q}{\partial t}-\frac{\alpha}{g}\frac{QB}{F^2}\frac{\partial H}{\partial t}+\left(1-\alpha\frac{Q^2B}{F^3g}\right)\frac{\partial H}{\partial x}+\alpha\frac{Q}{F^2g}\frac{\partial Q}{\partial x}=\frac{iK^2-Q^2}{K^2},$$

where Q – flow rate, F – cross-sectional area, B – width of the channel, H – depth, $K=CF\sqrt{R}$ – the capacity of the channel, C – Chezy coefficient, i – the angle of the midline of the channel bottom to the x-axis, α – parameter adjustments of movement; that in result of unperturbed established uniform motion is constant:

$$Q=Q_0,\ H=H_0,\ B=B_0,\ F=F_0,\ K=K_0.$$

As a result of perturbations:

$$H=H_0+\Delta H;\ Q=Q_0+\Delta Q;\ K=K_0+\Delta K;\ \ F=F_0+B_0\Delta H, \tag{2}$$

where $H_0,\ Q_0,\ K_0,\ F_0$ – constant values that are stored H, Q, K, F in a uniform flow.

Substituting the value of H, Q, K and F from (2) to the system (1):

$$\frac{\partial(Q_0+\Delta Q)}{\partial x}+\frac{\partial(F_0+B_0\Delta H)}{\partial t}=0;$$

$$\frac{1}{g}\frac{1}{F_0+B_0\Delta H}\frac{\partial(Q_0+\Delta Q)}{\partial t}-\frac{\alpha}{g}\frac{(Q_0+\Delta Q)B_0}{(F_0+B_0\Delta H)^2}\frac{\partial(H_0+\Delta H)}{\partial t}+$$
$$+\left(1-\alpha\frac{(Q_0+\Delta Q)^2 B_0}{(F_0+B_0\Delta H)^3 g}\right)\frac{\partial(H_0+\Delta H)}{\partial x}+$$
$$+\alpha\frac{(Q_0+\Delta Q)}{(F_0+B_0\Delta H)^2 g}\frac{\partial(Q_0+\Delta Q)}{\partial x}=\frac{Q_0^2(K_0+\Delta K)^2-(Q_0+\Delta Q)^2K_0^2}{K_0^2(K_0+\Delta K)^2}. \quad (3)$$

Simplifying each of the components of this system, discarding small quantities of second and higher order derivatives of small quantities we obtain:

$$\frac{1}{g}\frac{1}{F_0+B_0\Delta H}\frac{\partial(Q_0+\Delta Q)}{\partial t}=\frac{1}{g}\frac{1}{F_0}\frac{\partial\Delta Q}{\partial t},$$

$$\frac{\alpha}{g}\frac{(Q_0+\Delta Q)B_0}{(F_0+B_0\Delta H)^2}\frac{\partial(H_0+\Delta H)}{\partial t}=\frac{\alpha}{g}\frac{Q_0B_0}{F_0^2}\frac{\partial\Delta H}{\partial t},$$

$$\left(1-\alpha\frac{(Q_0+\Delta Q)^2 B_0}{(F_0+B_0\Delta H)^3 g}\right)\frac{\partial(H_0+\Delta H)}{\partial x}=\left(1-\frac{\alpha}{g}\frac{Q_0^2B_0}{F_0^3}\right)\frac{\partial\Delta H}{\partial x},$$

$$\alpha\frac{(Q_0+\Delta Q)}{(F_0+B_0\Delta H)^2 g}\frac{\partial(Q_0+\Delta Q)}{\partial x}=\frac{\alpha}{g}\frac{Q_0F_0}{F_0^3}\frac{\partial\Delta Q}{\partial x},$$

$$\frac{Q_0^2(K_0+\Delta K)^2-(Q_0+\Delta Q)^2K_0^2}{K_0^2(K_0+\Delta K)^2}=\frac{2\Delta KQ_0^2}{K_0^3}-\frac{2\Delta QQ_0}{K_0^2},$$

$$\frac{\partial(Q_0+\Delta Q)}{\partial x}=\frac{\partial Q_0}{\partial x}+\frac{\partial\Delta Q}{\partial x}=\frac{\partial\Delta Q}{\partial x},$$

$$\frac{\partial(F_0+B_0\Delta H)}{\partial t}=\frac{\partial F_0}{\partial t}+\frac{\partial(B_0\Delta H)}{\partial t}=\frac{\partial(B_0\Delta H)}{\partial t}.$$

Let the new variables $s=\frac{B}{F}x;\ \tau=t\sqrt{\frac{gB_0}{F_0}}$, and denote $u=\frac{\Delta Q}{Q_0};\ h=\frac{B_0\Delta H}{F_0}$, then the system (3) will look like:

$$\sqrt{\frac{Q_0^2B_0}{F_0^3g}}\frac{\partial u}{\partial s}+\frac{\partial h}{\partial\tau}=0;$$

$$\sqrt{\frac{Q_0^2B_0}{F_0^3g}}\frac{\partial u}{\partial\tau}-\alpha\sqrt{\frac{Q_0^2B_0}{F_0^3g}}\frac{\partial h}{\partial\tau}+\left(1-\alpha\frac{Q_0^2B_0}{F_0^3g}\right)\frac{\partial h}{\partial s}+\alpha\frac{Q_0^2B_0}{F_0^3g}\frac{\partial u}{\partial s}=\frac{2Q_0^2\Delta K}{K_0^2K_0}-\frac{2Q_0\Delta Q}{K_0^2}.$$

To simplify let's ignore the index zero at B, F, Q, K.

Let $\lambda = \sqrt{\frac{Q^2 B}{F^3 g}}$; $\mu = \frac{\Delta KF}{BK\Delta H}$; $i = \frac{Q^2}{K^2}$, then

$$\lambda \frac{\partial u}{\partial s} + \frac{\partial h}{\partial \tau} = 0;$$

$$\lambda \frac{\partial u}{\partial \tau} - \alpha\lambda \frac{\partial h}{\partial \tau} + \left(1 - \alpha\lambda^2\right) \frac{\partial h}{\partial s} + \alpha\lambda^2 \frac{\partial u}{\partial s} = 2i\left(\mu h - u\right). \tag{4}$$

From the first equation of (4) it follows that there exists a function $\Phi(s, \tau)$ that

$$h = \frac{\partial \Phi}{\partial s},\; u = -\frac{1}{\lambda}\frac{\partial \Phi}{\partial \tau}. \tag{5}$$

Substituting h and u in the second equation of system (4), we obtain:

$$\left(1 - \alpha\lambda^2\right)\frac{\partial^2 \Phi}{\partial s^2} - \frac{\partial^2 \Phi}{\partial \tau^2} - 2\alpha\lambda \frac{\partial^2 \Phi}{\partial s \partial \tau} = 2i\mu \frac{\partial \Phi}{\partial s} + \frac{2i}{\lambda}\frac{\partial \Phi}{\partial \tau}. \tag{6}$$

Putting in (6) $\Phi = e^{qs+r\tau}$, where q and r – constant, we obtain the equation:

$$\left(\alpha\lambda^2 - 1\right) q^2 + r^2 + 2r\left(\alpha\lambda q + \frac{i}{\lambda}\right) + 2i\mu q = 0. \tag{7}$$

To investigate the stability under perturbation of the boundary conditions take $r = \omega\sqrt{-1}$. Then equation (7) can be written as

$$\left(\alpha\lambda^2 - 1\right) q^2 - \omega^2 + 2\omega\left(\alpha\lambda q + \frac{i}{\lambda}\right)\sqrt{-1} + 2i\mu q = 0. \tag{8}$$

For stability it is necessary that condition is executed: Re q<0.

Applying Hurwitz theorem [10] to equation (8), we obtain

$$2\omega\frac{i}{\lambda} + 2i\mu q + q^2\left(\alpha\lambda^2 - 1\right)\sqrt{-1} + \omega^2\sqrt{-1} + 2\omega\alpha\lambda q\sqrt{-1} = 0, \tag{9}$$

where

$$a_0 = \left(\alpha\lambda^2 - 1\right);\; a_1 = 2\omega\alpha\lambda;\; a_2 = \omega^2;\; b_0 = 0;\; b_1 = 2i\mu;\; b_2 = 2\omega\frac{i}{\lambda}.$$

Using [10], we calculate the determinants

$$\nabla_2 = \begin{vmatrix} \alpha\lambda^2 - 1 & 2\omega\alpha\lambda \\ 0 & 2i\mu \end{vmatrix} = 2i\mu\left(\alpha\lambda^2 - 1\right);$$

$$\nabla_4 = \begin{vmatrix} \alpha\lambda^2 - 1 & 2\omega\alpha\lambda & \omega^2 & 0 \\ 0 & 2i\mu & 2\omega\frac{i}{\lambda} & 0 \\ 0 & \alpha\lambda^2 - 1 & 2\omega\alpha\lambda & \omega^2 \\ 0 & 0 & 2i\mu & 2\omega\frac{i}{\lambda} \end{vmatrix} = 4i^2\omega^2\left(\alpha\lambda^2 - 1\right)\left(\frac{1}{\lambda^2} - \left(\mu^2 - 2\alpha\mu + \alpha\right)\right).$$

Since that i=$\frac{Q^2}{K^2}$>0, take $\mu = \frac{\Delta KF}{BK\Delta H}$>0 and $\alpha\lambda^2 > 1$ then $\nabla_2 > 0$. We obtain the stability condition $\nabla_4 > 0$, that is $\frac{1}{\lambda^2} > \mu^2 - 2\alpha\mu + \alpha$.

Let's investigate the conditions of stability at perturbation of the initial conditions. For this, we assume $q = \omega\sqrt{-1}$ and get:

$$-\left(\alpha\lambda^2 - 1\right)\omega^2 + r^2 + 2r\left(\alpha\lambda\omega\sqrt{-1} + \frac{i}{\lambda}\right) + 2i\mu\omega\sqrt{-1} = 0.$$

Applying Hurwitz theorem [10], we have

$$\sqrt{-1}\left(\alpha\lambda^2 - 1\right)\omega^2 + \sqrt{-1}r^2 + 2r\left(\alpha\lambda\omega\sqrt{-1} + \frac{i}{\lambda}\right) + 2i\mu\omega = 0,$$

where

$$a_0 = 1;\ a_1 = 2\omega\alpha\lambda;\ a_2 = \left(\alpha\lambda^2 - 1\right)\omega^2;\ b_0 = 0;\ b_1 = 2\frac{i}{\lambda};\ b_2 = 2i\mu\omega.$$

We write the determinant

$$\nabla_2 = \begin{vmatrix} 1 & 2\omega\alpha\lambda \\ 0 & 2\frac{i}{\lambda} \end{vmatrix} = \frac{2i}{\lambda};$$

$$\nabla_4 = \begin{vmatrix} 1 & 2\omega\alpha\lambda & \omega^2\left(\alpha\lambda^2 - 1\right) & 0 \\ 0 & 2\frac{i}{\lambda} & 2i\mu\omega & 0 \\ 0 & 1 & 2\omega\alpha\lambda & \omega^2\left(\alpha\lambda^2 - 1\right) \\ 0 & 0 & 2\frac{i}{\lambda} & 2i\mu\omega \end{vmatrix} = 4i^2\omega^2\left(\frac{1}{\lambda^2} - \left(\mu^2 - 2\alpha\mu + \alpha\right)\right).$$

Let's estimate $\nabla_2 = \frac{2i}{\lambda}$.

Based on the fact that $i = \frac{Q^2}{K^2} > 0;\ \lambda = \sqrt{\frac{Q^2B}{F^3g}} > 0$, thus $\nabla_2 > 0$.

Then we have the only condition of stability $\nabla_4 > 0$, or $\frac{1}{\lambda^2} > \mu^2 - 2\alpha\mu + \alpha$.

Theorem 1. *Let* $\mu > 0,\ \alpha\lambda^2 > 1$, *then the system (1) is stable if the following condition holds:*

$$\frac{1}{\lambda^2} > \mu^2 - 2\alpha\mu + \alpha. \tag{10}$$

Condition (10) was first considered in [8] as necessary and sufficient condition for S-stability with respect to small perturbations, which takes into account the forces that affect the change in free surface flow. But then it was proved that this condition is only a necessary condition for S-stability. This criterion with $\alpha = 1$ was independently obtained by two authors [9, 10]. This criterion with $\alpha = 1$ in (10) becomes the following condition:

$$\frac{1}{\lambda} > \mu - 1. \tag{11}$$

Thus, having the condition $\operatorname{Re} q - \operatorname{Re} p\frac{\operatorname{Im} q}{\operatorname{Im} p} < 0$ [8] for the stability of channel flow we confirmed the correctness of the mathematical formulation (1) for the formulation of the problem of water flow motion in the pseudoprismatic channel.

3. Results of research

3. 1. Test of stability criterion. Test examples

Let's verify the stability condition for one-dimensional model of channel flow by changing the angle of the middle bottom line. **Table 1** shows the results for different values of α, yellow color – value at which the stability criterion is satisfied.

Table 1

Test of stability criterion for different bottom flow

№	L, m	B, m	H, m	U, m/s	U_0, m/s	F, m^2	$1/\lambda^2$	μ	α'	i	Ω^2	
											α=1	$\alpha=\alpha'$
1	150	2.5	0.04	10.2	4.2	0.1	0.0038	1.609	1.01	0.603	0.3709	0.3487
2	60	1.8	0.08	7.15	4.78	0.144	0.0153	1.553	1.027	0.243	0.305	0.2489
3	1160	1.8	0.3	9.26	7.2	0.54	0.03429	1.522	1.033	0.212	0.2725	0.205
4	600	1.9	0.107	6.94	5.5	0.2033	0.021772	1.574	1.04	0.148	0.3295	0.2435
5	200	2.6	0.06	5.32	4	0.156	0.020776	1.612	1.013	0.136	0.3745	0.4356
6	1158	4	0.08	6.89	5.7	0.32	0.016515	1.601	1.02	0.123	0.3612	0.3171
7	90	0.5	0.06	3.41	2.4	0.03	0.050567	1.476	1.05	0.104	0.2265	0.12897
8	176.5	0.5	0.06	2.94	2.28	0.03	0.068027	1.463	1.05	0.068	0.2144	0.11807
9	220	1.42	0.1	2.34	1.8	0.142	0.17897	1.564	1.12	0.055	0.3181	0.0627
10	220	1.42	0.1	2.34	1.8	0.142	0.17897	1.564	1.04	0.055	0.3181	0.23298
11	93	0.5	0.1	3.31	2.9	0.05	0.08944	1.408	1.05	0.043	0.1664	0.07567
12	240	0.5	0.08	2.76	2.42	0.04	0.10292	1.442	1.05	0.035	0.1954	0.10116
13	480	1.45	0.14	1.802	1.47	0.203	0.422517	1.564	1.080	0.034	0.3181	0.1479
14	460	1.7	0.12	1.68	1.4	0.204	0.4167	1.606	1.047	0.029	0.3672	0.26327
15	30	0.5	0.15	2.27	2.1	0.075	0.285276	1.391	1.05	0.017	0.1529	0.06378

Note: $\Omega^2 = \mu^2 - 2\alpha\mu + \alpha$, *values* μ *are taken from [7]; L – length of channel*

In **Fig. 1, 2** it can be seen the value of the left and right sides of the stability criterion for different values of α.

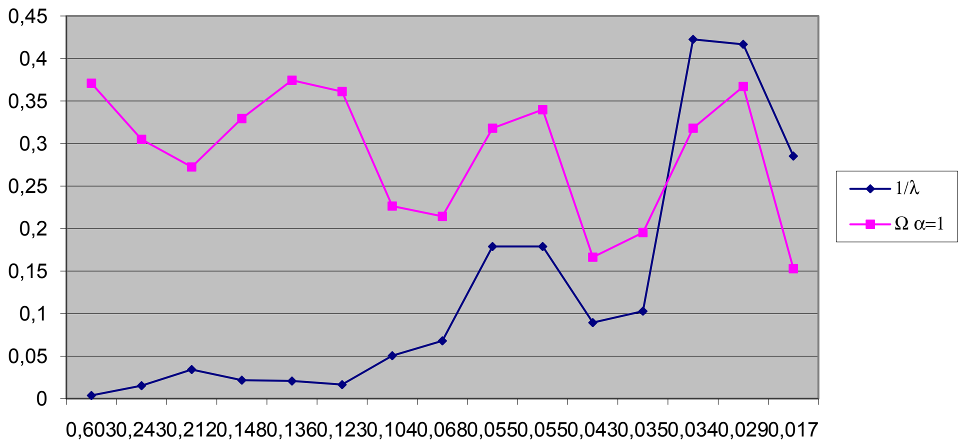

Fig. 1. The value of stability criterion for $\alpha = 1$

Fig. 1, 2 show execution of stability criterion (10) of Theorem 1 for different values of α and the values of parameter $1/\lambda$.

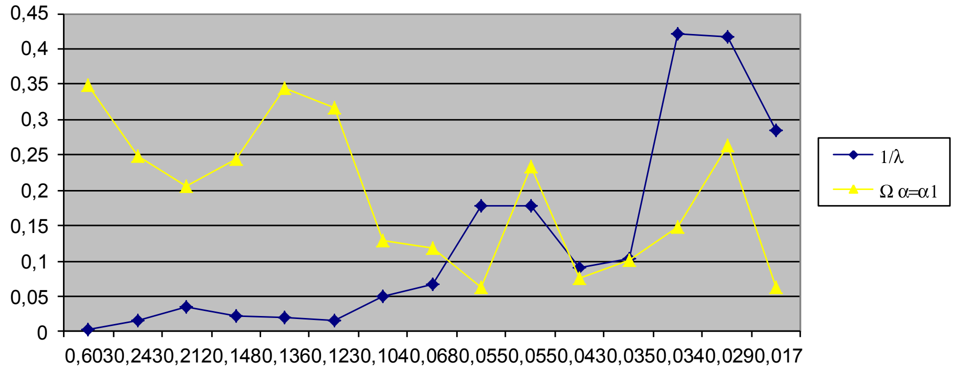

Fig. 2. The value of stability criterion for $\alpha = \alpha'$

4. Conclusions

In this article the main natural phenomena are considered, which may influence on wave stability of established flow in pseudoprismatic channels. It is shown that in this formulation of the problem we can consider only the impact of increasing amplitude on the wave. Under such conditions it should be considered not classical notion of stability of problem solution according to Lyapunov (L-stability), but proposed S-stability criterion for problem (1) of established flows in open pseudoprismatic channels, which analyzes the behavior of the wave amplitude for some time. Perturbation of initial and boundary conditions is considered to investigate the stability.

The system of equations [5] is investigated to describe the established channel flow. As a result of transformations we obtained equation, the real part of which solution should be negative. Hurwitz theorem is applied for determination of the sign. Constraints for determinants, which were formed from the coefficients of this equation, are derived on the basis of this theorem. Theorem 1 is formulated, in which stability criterion of the problem is given in specified conditions. Verification of criterion was conducted on test examples taken from laboratory investigations. The results show that decrease of the slope angle and increase of value α contributes to the fulfillment of stability criterion of the problem.

References

[1] Kuchment, L. S. (1980). Models of processes of formation of river flow. Gidrometeoizdat, 142.

[2] Frankl, F. I. (1955). On the theory of the movement of suspended sediment. Faculty of Kyrgyz State University, 3, 103–118.

[3] Vojnovich, V. A., Schwartz, A. I. (1946). A uniform motion of aerated water flows. VNIIG, 31.

[4] Venherskyi, P., Trushevskyi, V. (2014). Numerical modelling of shallow water flow in hydrodynamic approximations. Journal of Computational and Applied Mathematics, 2 (116), 152–166.

[5] Venherskyi, P. S., Kokovska, Y. V. (2009). About mathematical models fluid flow channel. Dynamical System Modeling and Stability Investigation: "DSMSI-2009", 119–120.

[6] Venherskyi, P., Kokovska, Y. (2014) Numerical Modeling of Water Movement in the River Network in thr Selected, Modern problems of radio engineering, telecommunications and computer science, TCSET'2014, 83–85.

[7] Kartvelishvili, N. A. (1973). Streams in non-deformable beds. Gidrometeoizdat, 279.

[8] Kartvelishvili, N. A. (1968). Unsteady open streams. Gidrometeoizdat, 126.

[9] Vedernikov, V. V., Mastitsky, N. V., Potapov, M. V. (1947). Unsteady motion of water flow in open channels. USSR Academy of Sciences, 89.

[10] Iwasa, Y. (1954). The criterion for instability of steady uniform flows in open channels. Mem. Fac. Eng. Kyoto Univ., 16 (4), 811–817.